Fundamentals of Reliability Engineering and Applications

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Quality Control & Reliability Engineering (QCRE)
IIE

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Outline
Part 1. Reliability Definitions

- Reliability Definition…Time dependent characteristics
- Failure Rate
- Availability
- MTTF and MTBF
- Time to First Failure
- Mean Residual Life
- Conclusions
Outline

Part 2. Reliability Calculations

1. Use of failure data
2. Density functions
3. Reliability function
4. Hazard and failure rates
Outline
Part 3. Failure Time Distributions

1. Constant failure rate distributions
2. Increasing failure rate distributions
3. Decreasing failure rate distributions
4. Weibull Analysis – Why use Weibull?
Outline
Part 2. Reliability Calculations

1. Use of failure data
   a) Interval data (no censoring)
   b) Exact failure times are known
2. Density functions
3. Reliability function
4. Hazard and failure rates
Basic Calculations

Suppose $n_0$ identical units are subjected to a test. During the interval $(t, t + \Delta t)$, we observed $n_f(t)$ failed components. Let $n_s(t)$ be the surviving components at time $t$, then we define:

- **Failure density function**
  
  $$f(t) = \frac{n_f(t)}{n_0 \Delta t}$$

- **Failure rate function**
  
  $$\hat{h}(t) = \frac{n_f(t)}{n_s(t) \Delta t}$$

- **Reliability function**
  
  $$\hat{R}(t) = P_r(T > t) = \frac{n_s(t)}{n_0}$$
The unreliability $F(t)$ is

$$F(t) = 1 - R(t)$$

**Example:** 200 light bulbs were tested and the failures in 1000-hour intervals are

<table>
<thead>
<tr>
<th>Time Interval (Hours)</th>
<th>Failures in the interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1000</td>
<td>100</td>
</tr>
<tr>
<td>1001-2000</td>
<td>40</td>
</tr>
<tr>
<td>2001-3000</td>
<td>20</td>
</tr>
<tr>
<td>3001-4000</td>
<td>15</td>
</tr>
<tr>
<td>4001-5000</td>
<td>10</td>
</tr>
<tr>
<td>5001-6000</td>
<td>8</td>
</tr>
<tr>
<td>6001-7000</td>
<td>7</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>200</strong></td>
</tr>
</tbody>
</table>
## Calculations

<table>
<thead>
<tr>
<th>Time Interval (Hours)</th>
<th>Failures in the interval</th>
<th>Failure Density $f(t) \times 10^{-4}$</th>
<th>Hazard rate $h(t) \times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1000</td>
<td>100</td>
<td>$\frac{100}{200 \times 10^3} = 5.0$</td>
<td>$\frac{100}{200 \times 10^3} = 5.0$</td>
</tr>
<tr>
<td>1001-2000</td>
<td>40</td>
<td>$\frac{40}{200 \times 10^3} = 2.0$</td>
<td>$\frac{40}{100 \times 10^3} = 4.0$</td>
</tr>
<tr>
<td>2001-3000</td>
<td>20</td>
<td>$\frac{20}{200 \times 10^3} = 1.0$</td>
<td>$\frac{20}{60 \times 10^3} = 3.33$</td>
</tr>
<tr>
<td>3001-4000</td>
<td>15</td>
<td>......................................</td>
<td>...................................</td>
</tr>
<tr>
<td>4001-5000</td>
<td>10</td>
<td>......................................</td>
<td>...................................</td>
</tr>
<tr>
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<td>7</td>
<td>......................................</td>
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<td><strong>Total</strong></td>
<td><strong>200</strong></td>
<td>......................................</td>
<td>...................................</td>
</tr>
</tbody>
</table>
Failure Density vs. Time

The graph shows the failure density function $f(t)$ for different times in hours. The x-axis represents time in hours, ranging from $1 \times 10^3$ to $7 \times 10^3$, and the y-axis represents $f(t)$ in units of $10^{-4}$. The chart indicates the distribution of failures over time.
Hazard Rate vs. Time

![Hazard Rate vs. Time Graph]
# Reliability Calculations

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<td>7</td>
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<tr>
<td><strong>Total</strong></td>
<td><strong>200</strong></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>Reliability $R(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–1000</td>
<td>5/5 = 1.0</td>
</tr>
<tr>
<td>1001–2000</td>
<td>2.0/4.0 = 0.5</td>
</tr>
<tr>
<td>2001–3000</td>
<td>1/3.33 = 0.33</td>
</tr>
<tr>
<td>......</td>
<td>......</td>
</tr>
<tr>
<td>6001–7000</td>
<td>0.35/10 = .035</td>
</tr>
</tbody>
</table>
Reliability vs. Time

![Graph showing reliability over time](image)

- **Reliability ($R(t)$)**: Y-axis ranging from 0 to 1.2.
- **Time in hours**: X-axis ranging from 1 to 7 x 10^3.

The graph illustrates the decline in reliability over time, with a peak at 1 x 10^3 hours and a steady decrease thereafter.
Exponential Distribution

**Definition**

\[ h(t) = \lambda \quad \lambda > 0, \quad t \geq 0 \]

\[ f(t) = \lambda \exp(-\lambda t) \]

\[ R(t) = \exp(-\lambda t) = 1 - F(t) \]
Exponential Model Cont’d

**Statistical Properties**

\[
MTTF = \frac{1}{\lambda} \quad \lambda = 5 \times 10^{-6} \text{ Failures/hr}
\]

MTTF = 200,000 hrs or 20 years

\[
\text{Variance} = \frac{1}{\lambda^2} \quad \lambda = 5 \times 10^{-6} \text{ Failures/hr}
\]

Standard deviation of MTTF is 200,000 hrs or 20 years

\[
\text{Median life} = (\ln 2) \frac{1}{\lambda} \quad \text{Median life} = 138,626 \text{ hrs or 14 years}
\]
Exponential Model Cont’d

Statistical Properties

\[ MTTF = \frac{1}{\lambda} \]

\[ \lambda = 5 \times 10^{-6} \text{ Failures/hr} \]

MTTF=200,000 hrs or 20 years

It is important to note that the MTTF= \(1/\text{failure rate}\) is only applicable for the constant failure rate case (failure time follow exponential distribution.)
Empirical Estimate of $F(t)$ and $R(t)$

When the exact failure times of units is known, we use an empirical approach to estimate the reliability metrics. The most common approach is the Rank Estimator. Order the failure time observations (failure times) in an ascending order:

$$t_1 \leq t_2 \leq \ldots \leq t_{i-1} \leq t_i \leq t_{i+1} \leq \ldots \leq t_{n-1} \leq t_n$$
Empirical Estimate of $F(t)$ and $R(t)$

$F(t_i)$ is obtained by several methods

1. Uniform “naive” estimator \( \frac{i}{n} \)

2. Mean rank estimator \( \frac{i}{n+1} \)

3. Median rank estimator (Bernard) \( \frac{i - 0.3}{n + 0.4} \)

4. Median rank estimator (Blom) \( \frac{i - 3/8}{n + 1/4} \)
Empirical Estimate of $F(t)$ and $R(t)$

Assume that we use the mean rank estimator

$$
\hat{F}(t_i) = \frac{i}{n+1}
$$

$$
\hat{R}(t_i) = \frac{n+1-i}{n+1} \quad t_i \leq t \leq t_{i+1} \quad i = 0, 1, 2, ..., n
$$

Since $f(t)$ is the derivative of $F(t)$, then

$$
\hat{f}(t_i) = \frac{\hat{F}(t_{i+1}) - \hat{F}(t_i)}{\Delta t_i \cdot (n+1)} \quad \Delta t_i = t_{i+1} - t_i
$$

$$
\hat{f}(t_i) = \frac{1}{\Delta t_i \cdot (n+1)}
$$
Empirical Estimate of $F(t)$ and $R(t)$

\[
\hat{\lambda}(t_i) = \frac{1}{\Delta t_i \cdot (n + 1 - i)}
\]

\[
\hat{H}(t_i) = -\ln(\hat{R}(t_i))
\]

Example:

Recorded failure times for a sample of 9 units are observed at $t=70, 150, 250, 360, 485, 650, 855, 1130, 1540$. Determine $F(t), R(t), f(t), \lambda(t), H(t)$.
# Calculations

<table>
<thead>
<tr>
<th>i</th>
<th>t (i)</th>
<th>t(i+1)</th>
<th>F=i/10</th>
<th>R=(10-i)/10</th>
<th>f=0.1/Δt</th>
<th>λ =1/(Δt.(10-i))</th>
<th>H(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>70</td>
<td>0</td>
<td>1</td>
<td>0.001429</td>
<td>0.001429</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>70</td>
<td>150</td>
<td>0.1</td>
<td>0.9</td>
<td>0.001250</td>
<td>0.001389</td>
<td>0.10536052</td>
</tr>
<tr>
<td>2</td>
<td>150</td>
<td>250</td>
<td>0.2</td>
<td>0.8</td>
<td>0.001000</td>
<td>0.001250</td>
<td>0.22314355</td>
</tr>
<tr>
<td>3</td>
<td>250</td>
<td>360</td>
<td>0.3</td>
<td>0.7</td>
<td>0.000909</td>
<td>0.001299</td>
<td>0.35667494</td>
</tr>
<tr>
<td>4</td>
<td>360</td>
<td>485</td>
<td>0.4</td>
<td>0.6</td>
<td>0.000800</td>
<td>0.001333</td>
<td>0.51082562</td>
</tr>
<tr>
<td>5</td>
<td>485</td>
<td>650</td>
<td>0.5</td>
<td>0.5</td>
<td>0.000606</td>
<td>0.001212</td>
<td>0.69314718</td>
</tr>
<tr>
<td>6</td>
<td>650</td>
<td>855</td>
<td>0.6</td>
<td>0.4</td>
<td>0.000488</td>
<td>0.001220</td>
<td>0.91629073</td>
</tr>
<tr>
<td>7</td>
<td>855</td>
<td>1130</td>
<td>0.7</td>
<td>0.3</td>
<td>0.000364</td>
<td>0.001212</td>
<td>1.2039728</td>
</tr>
<tr>
<td>8</td>
<td>1130</td>
<td>1540</td>
<td>0.8</td>
<td>0.2</td>
<td>0.000244</td>
<td>0.001220</td>
<td>1.60943791</td>
</tr>
<tr>
<td>9</td>
<td>1540</td>
<td>-</td>
<td>0.9</td>
<td>0.1</td>
<td></td>
<td></td>
<td>2.30258509</td>
</tr>
</tbody>
</table>
Probability Density Function

![Graph of Probability Density Function](image)
Constant Failure Rate

![Graph showing constant failure rate over time.](image-url)
Exponential Distribution: Another Example

Given failure data:

Plot the hazard rate, if constant then use the exponential distribution with $f(t)$, $R(t)$ and $h(t)$ as defined before.

We use a software to demonstrate these steps.
Input Data

Detected sample size and classify data in ascending order

Sample Size  50
Failures     50

Complete Sample
Censored by Nr. Units
Censored by Time
Random Censoring

Location Parameter
- Provided by User
- Provided by Computer

User's Observations:
Test for all models, complete sample

Type 'C' to toggle between complete and censored time to failure
Type 'P' to reproduce the value of the previous cell
Plot of the Data
Exponential Fit

Model: \( f(t) = \lambda \exp(-\lambda t) \)

Where:
- \( \lambda \) - lambda parameter (failure rate)
- \( t \) - time
Exponential Analysis

**Probability Density**

- $f(t)$
- $0.000$ to $0.005$ on the y-axis
- Time / 100 on the x-axis

**Reliability**

- $R(t)$
- $1.0$ to $0.0$ on the y-axis
- Time / 100 on the x-axis

**Hazard Rate**

- $h(t)$
- $0.000$ to $0.005$ on the y-axis
- Time / 100 on the x-axis

**Cumulative Prob. of Failure**

- $F(t)$
- $0.0$ to $1.0$ on the y-axis
- Time / 100 on the x-axis
Summary

In this part, we presented the three most important relationships in reliability engineering.

We estimated obtained estimate functions for failure rate, reliability and failure time. We obtained these function for interval time and exact failure times.